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Note

The minimum forcing number for the torus and hypercube

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Abstract

Let G be a graph with a perfect matching M. Define the forcing number of M in G to be the smallest size of a subset $S \subset M$ that is in no other perfect matching. In this paper, we present a property of bipartite graphs G that acts as a lower bound on the forcing number of perfect matchings in G. We then apply this to the torus and the hypercube, proving that the minimum forcing number of a perfect matching on a $2m \times 2n$ torus with $m \geqslant n$ is $2n$, and that the minimum forcing number on an n-dimensional hypercube is $2^n/4$ if n is even.

Keywords: Forcing number; Perfect matching; Torus; Hypercube

1. Introduction

Definition. Let G be a graph and let M be a perfect matching of G, where a perfect matching is defined as a set of disconnected edges from G that cover all of G. A subset S of M is said to *force* M if S is in no other perfect matching. The *forcing number* of M is defined as the smallest number of edges in a subset S that forces M.

The notion of the forcing number of a matching was introduced by Harary et al. [2], having been motivated by problems in chemistry which reduce to determining forcing numbers in hexagonal systems (see for example [3,5]). The forcing number of square grids has also been studied in [4], but little has been done with forcing numbers of non-planar graphs.

Definition. An *alternating cycle* in a matching M of a graph G is a cycle in G in which the edges alternate between edges from M and edges not from M.

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It was shown in [4] that for planar bipartite graphs, the forcing number of a perfect matching is equal to the greatest number of disjoint alternating cycles in the matching. In graphs where this result holds, the idea of looking for disjoint alternating cycles provides a nice way of putting bounds on the forcing number. Unfortunately, it does not hold for many non-planar graphs, including the torus and hypercube, so for these graphs another approach is needed.

In this paper we present an alternative way of finding a lower bound on the forcing number in bipartite graphs. We simplify the problem by finding a property of the graph G that acts as a lower bound on the forcing number of any perfect matching on G; this property is easier to determine than the original formulation of the problem because it is independent of the perfect matching being considered. We then apply this result to the case of the torus and the hypercube, finding the minimum forcing number of a perfect matching on a $2m \times 2n$ torus, and on a hypercube of even dimension. These results resolve the conjecture of Pachter and Kim that the minimum forcing number for a $2n \times 2n$ torus is $2n$ [4], and in the case where n is even it resolves their conjecture that the minimum forcing number for an n-dimensional hypercube is $2^n/4$ [4].

2. Preliminaries

The following proposition is equivalent to a proposition from [4]:

Proposition 1. *If a subset S of M forces M then S contains an edge from every alternating cycle in M.*

Proof. If there were an alternating cycle C in M that did not intersect S, then you could shift the edges of C, giving a different perfect matching M' that still contains S. $\square$

In what follows, G will denote a bipartite graph with A and B its partite classes. Let $\alpha : E(G) \to A$ be the function that maps an edge e to its endpoint in A, and $\beta : E(G) \to B$ be the function that maps e to its endpoint in B. If S is a set of edges, $\alpha(S)$ and $\beta(S)$ will refer to the image of S under α and β.

Definition. For any subset S of a perfect matching M, we say that a vertex is *forced* by S if all but one of its neighbors are endpoints of some edge in S. An edge e is *A-forced* by S if $\alpha(e)$ is forced by S. If there exists a sequence of sets $S = S_0, S_1, S_2, \ldots, S_k$ and a sequence of edges $e_1, e_2, \ldots, e_k$ where $S_i = S_{i-1} \bigcup \{e_i\}$ and S_{i-1} A-forces e_i for each i, then we say that S *A-forces* the set S_k.

Lemma 1. *S A-forces M if and only if S forces M.*

Proof. It is clear from the definition that if S A-forces M, it must also force M. To prove the other direction, we will assume that S does not A-force M, and show that S does not force M.

Let $A(S)$ be the collection of all sets which are A-forced by S, and take $S_{\max}$ to be a maximal element of $A(S)$, so that no edges are A-forced by $S_{\max}$.

Notice that since S is a subset of M, and $S_{\max}$ is A-forced by S, $S_{\max}$ must also be a subset of M. We look at the graph that remains when $S_{\max}$ is removed, and show that it must contain an alternating cycle. Let $L = M \setminus S_{\max}$, and let $G(L)$ be the induced subgraph of G on the vertex set of L. Note that L is a perfect matching of $G(L)$.

Pachter and Kim [4] provide a method of constructing a directed graph $D(L)$ from a perfect matching L on a bipartite graph $G(L)$ that yields a one-to-one correspondence between alternating cycles in L and directed cycles in $D(L)$. This is accomplished by assigning orientations to the edges of $G(L)$ as follows: If the edge e is in L, orient e from $\beta(e)$ to $\alpha(e)$, otherwise orient it from $\alpha(e)$ to $\beta(e)$.

Since no edges are A-forced by $S_{\max}$, no vertices of A can have degree one in $G(L)$, so every vertex of A in $D(L)$ must have an out-degree of at least one. Furthermore, every vertex in $G(L)$ is incident to one edge of L, so the vertices of B in $D(L)$ have an out-degree of one. Therefore, we can find a directed cycle in $D(L)$ simply by following a path randomly until it hits itself. This cycle must correspond to an alternating cycle in $G(L)$, which does not intersect S, so by Proposition 1, S must not force M. $\square$

3. Lower bounds on the forcing number in bipartite graphs

At this point we shift our attention to the vertices in the partite class B, and see what we can say about $\beta(S)$ for forcing sets S. In this section we will find a lower bound on the size of $\beta(S)$ which applies to all perfect matchings of G, thus reducing a problem about all perfect matchings to a single problem about the vertices in B.

In order to preserve information about the structure of the graph, we look at the neighborhoods, $N(a)$ of vertices $a \in A$; $N(a)$ is defined to be the set of vertices that are adjacent to a.

Assign an ordering $e_1 < e_2 < \cdots < e_n$ to the edges of M, and let $b_1 < b_2 < \cdots < b_n$ be the unique ordering on the vertices of B such that $b_i = \beta(e_i)$ for all i. Then we make the following definition.

Definition. A vertex b *leads* a neighborhood $N(a)$ if b is the largest vertex in $N(a)$. A vertex b is called a *leading vertex* if it leads *at least one* set $N(a)$ for some $\in A$, and it is called a *trailing vertex* if it is not a leading vertex, i.e. if it does not lead any neighborhoods $N(a)$.

For notational convenience, let $E_i = \{e_1, e_2, \ldots, e_i\}$ and $B_i = \beta(E_i) = \{b_1, b_2, \ldots, b_i\}$. The following lemma will then hold.

Lemma 2. *If E_i A-forces e_{i+1}, then b_{i+1} leads the set $N(a_{i+1})$, where $a_{i+1} = \alpha(e_{i+1})$.*

Proof. The vertex a_{i+1} is forced by E_i, so only one of its neighbors is not in B_i. That neighbor must be b_{i+1}, and b_{i+1} is larger than every vertex in B_i, so b_{i+1} leads the set $N(a_{i+1})$. □

Now we are ready to state and prove a theorem giving a lower bound on the forcing number of perfect matchings in G.

Theorem 1. *The forcing number of a perfect matching of a bipartite graph G with partite classes A and B is bounded below by the minimum number of trailing vertices in B over all possible orderings of B.*

Proof. Let M be a perfect matching of G which has forcing number k, and contains n edges, and let S_k be a subset of M of size k that forces M. We will construct an ordering of the vertices of B so that every trailing vertex must be in $\beta(S_k)$, thus demonstrating that the minimum number of trailing vertices in B is at most k.

Since S_k forces M, by Lemma 1, S_k also A-forces M. Therefore, we can construct sets $S_k, S_{k+1}, S_{k+2}, \ldots, S_n = M$ and edges $e_{k+1}, e_{k+2}, \ldots, e_n$ with S_i A-forcing e_{i+1} and $S_{i+1} = S_i \cup \{e_{i+1}\}$ for all $i \geqslant k$. This yields an ordering of the edges $e_{k+1}, e_{k+2}, \ldots, e_n$ not in S_k. We complete the ordering by assigning the edges of S_k the labels $e_1, \ldots, e_k$. Then the sets $S_k, S_{k+1}, \ldots, S_n$ will be equal to the sets $E_k, E_{k+1}, \ldots, E_n$, so that every edge e_{i+1} not in S is A-forced by E_i. If we give the vertices of B the same order $b_1, b_2, \ldots, b_n$ as their associated edges, we can apply Lemma 2 to show that the vertices not in $\beta(S)$ are all leading vertices. □

We will use Theorem 1 to prove the minimality of our results for the torus and hypercube. In the case of the torus it will be useful to further modify the bound in Theorem 1 first, to get another lower bound on the forcing number which is not quite as strong as the bound given in Theorem 1, but in some cases it is easier to compute. First, we introduce some more terminology.

As before, assign an ordering $b_1 < b_2 < \cdots < b_n$ to B. Denote the set $\{b_{k+1}, b_{k+2}, \ldots, b_n\}$ by $\bar{B}_k$.

Definition. For a set of vertices T, let $N(T)$ be the set of all vertices adjacent to T; then define the *excess* of T as $e(T) = |N(T)| - |T|$. The *maximum excess* of an ordering of B is the maximum value of $e(\bar{B}_k)$ over all k.

Proposition 2. *A vertex b_k is a trailing vertex if and only if $e(\bar{B}_{k-1}) = e(\bar{B}_k) - 1$, and otherwise $e(\bar{B}_{k-1}) \geqslant e(\bar{B}_k)$.*

Proof. A vertex $a \in A$ is in $N(\bar{B}_{k-1})$ if and only if its neighborhood $N(a)$ contains a vertex from $\bar{B}_{k-1}$, which occurs if and only if the leading vertex of $N(a)$

is at least b_k. So the number of sets with leading vertex b_k is equal to $|N(\bar{B}_{k-1})| - |N(\bar{B}_k)| = e(\bar{B}_{k-1}) - e(\bar{B}_k) + 1$. Therefore b_k is a trailing vertex if and only if $e(\bar{B}_{k-1}) - e(\bar{B}_k) + 1 = 0$, and otherwise we have $e(\bar{B}_{k-1}) - e(\bar{B}_k) \geqslant 0$. □

Theorem 2. *The forcing number of any perfect matching on a bipartite graph is bounded below by the smallest possible maximum excess for all orderings of* B.

Proof. For a given ordering $b_1 < b_2 < \cdots < b_n$, let its maximum excess be x, and let $\bar{B}_k$ be a set for which $e(\bar{B}_k) = x$. Since $e(\bar{B}_0) = 0$, as the vertices $b_k, \ldots, b_1$ are added to $\bar{B}_k$, the excess decreases by a total of x. Therefore, by Proposition 2, at least x of the vertices that are added must be trailing vertices. Since this argument applies for all orderings, the smallest maximum excess is less than or equal to the minimum number of trailing vertices, so applying Theorem 1, we get the desired result. □

This bound is weaker than the bound given in Theorem 1, and, in fact, it is not hard to find a family of graphs for which the smallest maximum excess remains at one while the forcing number diverges to infinity—take, for example, the graph consisting of n disjoint four-cycles.

However in many graphs, such as the torus, the smallest maximum excess is equal to the forcing number, and for these graphs it may be easier to compute the maximum excess than the number of trailing vertices.

4. Lower bound for the $2n \times 2m$ torus

Now we apply this result to the case of the torus. The lower bound in question was conjectured for the case of a square torus by Pachter and Kim [4]; we prove and extend this conjecture.

Definition. A $2n \times 2m$ torus is defined as $C_{2n} \times C_{2m}$, where C_{2n} and C_{2m} are cycles of length $2n$ and $2m$, respectively, and $\times$ is the cartesian product for graphs. In what follows, we will call the $2m$ copies of C_{2n} in the cartesian product the rows of the torus, and the $2n$ copies of C_{2m} the columns of the torus.

Theorem 3. *The forcing number of a perfect matching on a* $2n \times 2m$ *torus with* $m \geqslant n$ *is at least* $2n$. *Furthermore, this bound is sharp.*

Proof. First, the example shown in Fig. 1 demonstrates that this bound can be achieved for a 4×6 torus. It is easy to see that this pattern can be extended to cover all m and n.

We will say that a set $T \subseteq B$ *fills* a row or column R if $R \subseteq T \cup N(T)$, and that T *touches* R if $(T \cup N(T)) \cap R \neq \emptyset$. Note that filling a row or column R is equivalent to containing $B \cap R$.

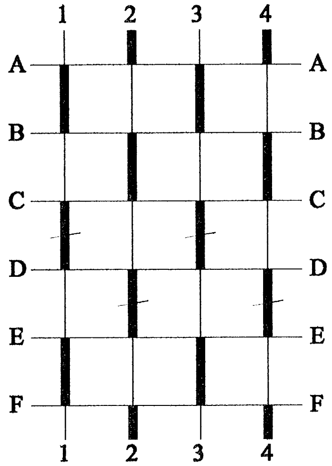

Fig. 1. A perfect matching with minimum forcing number on a 4×6 torus. The edges labeled with the same letter or number are identified, and the marked edges force the matching.

For any ordering $b_1 < b_2 < \cdots < b_{2mn}$ of B, we again denote the set $\{b_{i+1}, b_{i+2}, \ldots, b_{2mn}\}$ by $\bar{B}_i$. Take the smallest set $\bar{B}_k$ which fills some row or column. We will show that either $e(\bar{B}_k) \geqslant 2n$, or $e(\bar{B}_{k+1}) \geqslant 2n$.

We first note the following facts:

Fact 1. *If $\bar{B}_k$ fills a row (or column), then $\bar{B}_k$ touches every column (or row).*

Fact 2. *Every row or column that $\bar{B}_k$ touches but does not fill contains more points from $N(\bar{B}_k)$ than from $\bar{B}_k$.*

Now let us look at a few cases:

Case 1: $\bar{B}_k$ fills a row but not a column. Then by Fact 1, $\bar{B}_k$ touches every column. But by hypothesis it fills no columns, so by Fact 2, $N(\bar{B}_k)$ must contain at least $2n$ more vertices than $\bar{B}_k$ (one from each column). So $e(\bar{B}_k) \geqslant 2n$.

The case where $\bar{B}_k$ fills a column but not a row is identical, yielding $e(\bar{B}_k) \geqslant 2m$ which yields $e(\bar{B}_k) \geqslant 2n$ as desired, since $2m \geqslant 2n$.

Case 2: $\bar{B}_k$ fills both a row and a column. In this case, consider $\bar{B}_{k+1}$. Since $\bar{B}_{k+1}$ does not fill any rows or columns, the vertex b_{k+1} that was removed from $\bar{B}_k$ was taken from a full column. Therefore, unless $m = n = 1$, every column is still touched by $\bar{B}_{k+1}$ (it is easy to prove the lower bound separately for the trivial case $m = n = 1$). Again we have $2n$ columns which are touched but not filled by $\bar{B}_{k+1}$, so by Fact 2, $e(\bar{B}_{k+1}) \geqslant 2n$.

This completes the proof that any ordering has maximum excess at least $2n$. Applying Theorem 2, we conclude that the forcing number of any perfect matching of the $2m \times 2n$ torus is at least $2n$. $\square$

This method can also be used to give an alternate proof that the lower bound of the forcing number of a matching on a $2n \times 2n$ square grid is n, as proved by Pachter and Kim [4].

The only difference in the proof is that Fact 2 needs to be modified. On a square grid, a row (or column) that is touched but not filled does not necessarily have a positive excess, but it can be shown that any two adjacent columns which are both touched but not filled must together have excess at least one. This leads to a lower bound, n, which is half of the number of columns or rows.

5. Lower bound for the hypercube

Let Q_n denote the hypercube of dimension n. It was conjectured in [4] that the minimum forcing number of a perfect matching on Q_n is $2^n/4$. Using Theorem 1 we can prove this for the case when n is even.

It should be noted that the proof cannot use Theorem 2, since there are orderings of the vertices of Q_n for which the maximum excess is less than the desired lower bound of $2^n/4$.

The smallest hypercube for which the smallest maximum excess is less than $2^n/4$ is the 5-cube, which has smallest maximum excess of 7. A 6-cube can have maximum excess as low as 13, and a 7-cube can have maximum excess as low as 23.

These low maximum excesses can be achieved by putting all the vertices of Q_n in a sequence as follows: Choose any vertex to be the first term in the sequence, and let its neighbors be the next n terms, in any order. Each successive term is determined by finding the first vertex in the sequence which has neighbors that are not yet in the sequence, and choosing one of those neighbors to add to the end of the sequence. Eventually, every vertex of Q_n will be added. The ordering of the vertices in one of the partite classes that is determined by the order they occur in this sequence will have the desired low maximum excess.

Since we cannot use the concept of maximum excess, we will instead have to prove directly that any ordering of Q_n must have at least $2^n/4$ trailing vertices. We first introduce some notation and prove a lemma that will be used in the proof.

Let A and B be the partite classes of the hypercube. For any vertex v and set of vertices T in G, let $\deg(v, T)$ denote the number of edges connecting v to some vertex in T. For sets $Q \subseteq A$ let $N_{\text{odd}}(Q)$ be the set of all $\in B$ such that $\deg(b, Q)$ is odd.

Lemma 3. *For any $Q \subseteq A$ in a hypercube of even dimension, $N_{\text{odd}}(Q)$ is either empty or it contains a trailing vertex.*

Proof. Suppose that $N_{\text{odd}}(Q)$ is non-empty and does not contain any trailing vertices. Then the smallest vertex b of $N_{\text{odd}}(Q)$ must lead some neighborhood $N(a)$, and must therefore be the only vertex from $N_{\text{odd}}(Q)$ in $N(a)$.

Since $\deg(b,Q)$ is odd exactly when $b \in N_{\text{odd}}(Q)$, there must be exactly one odd term in the sum $\sum_{b\in N(a)} \deg(b,Q)$, so the sum itself must be odd. Since both sums $\sum_{b\in N(a)} \deg(b,Q)$ and $\sum_{q\in Q} \deg(q,N(a))$ count the number of edges connecting the sets $N(a)$ and Q, the sum $\sum_{q\in Q} \deg(q,N(a))$ must also be odd.

We obtain a contradiction by showing that for every $q \in Q$, $\deg(q,N(a))$ is in fact even. Because $\deg(q,N(a))$ counts the number of vertices adjacent to both a and q, in order for it to be non-zero the distance between a and q must be either 0 or 2. If $a = q$, then there are n vertices adjacent to a and q, and n is even since we are dealing with a hypercube of even dimension. If a and q are distance 2 apart, then they must be the opposite corners of some square. The only points adjacent to both a and q will be the other two corners of the square, so $\deg(q,N(a)) = 2$. This proves that for any a and q, $\deg(q,N(a))$ must be even, which demonstrates that our earlier conclusion that $\sum_{q\in Q} \deg(q,N(a))$ is odd must have been false. Therefore, the assumption that $N_{\text{odd}}(Q)$ was non-empty and contained no trailing vertices must have been false as well. □

We are now ready to prove the following theorem:

Theorem 4. *For even n, the forcing number of any perfect matching on Q_n is at least $2^n/4$. Furthermore, this bound is sharp.*

Proof. First, we show that this bound can be achieved by dividing Q_n into two sub-cubes H_0 and H_1 of dimension $n-1$, and taking the perfect matching M that consists of all the edges that join H_0 to H_1. If we let S be the set of edges $e \in M$ for which $\beta(e)$ is in H_0, then every vertex not covered by S is forced, so S forces M.

To prove the lower bound we will show that every set S of size $2^n/4+1$ must contain a trailing vertex in every ordering of B. This will imply that B contains at least $2^n/4$ trailing vertices, and therefore, by Theorem 1, that the minimum forcing number must be at least $2^n/4$.

Since any non-empty set of the form $N_{\text{odd}}(Q)$ contains a trailing vertex, we need only find a set Q for which $N_{\text{odd}}(Q)$ is a non-empty subset of S. We again divide Q_n into sub-cubes H_0 and H_1 of dimension $n-1$, and let $A_0 = A \cap H_0$. We will restrict our search for Q to subsets of A_0 to ensure that $N_{\text{odd}}(Q)$ will not be empty. Define a function $F : 2^{A_0} \to 2^{B\setminus S}$ by $F(T) = N_{\text{odd}}(T) \cap (B \setminus S)$ for any $T \subseteq A_0$. Note that $F(T) = \emptyset$ exactly when $N_{\text{odd}}(T) \subseteq S$. If $|S| = 2^n/4+1$, then 2^{A_0} is larger than $2^{B\setminus S}$, so by the pigeon hole principle, we can find two subsets T_1 and T_2 of A_0 for which $F(T_1) = F(T_2)$. We leave it to the reader to show that F satisfies the linearity condition $F(T_1 \Delta T_2) = F(T_1) \Delta F(T_2)$, where $S_1 \Delta S_2$ denotes the symmetric difference $(S_1 \cup S_2) - (S_1 \cap S_2)$. So if we let $Q = T_1 \Delta T_2$, we have $F(Q) = F(T_1) \Delta F(T_2) = \emptyset$, so $N_{\text{odd}}(Q)$ lies entirely in S. Note that $N_{\text{odd}}(Q)$ is non-empty because the neighbors of Q in H_1 are adjacent to exactly one vertex of Q and are therefore in $N_{\text{odd}}(Q)$. So by Lemma 3, $N_{\text{odd}}(Q)$ contains a trailing vertex, and therefore S contains a trailing vertex, which completes the proof. □

6. Other problems

One natural extension of these results would be to completely resolve the conjecture of Pachter and Kim that the minimum forcing number for an n-dimensional hypercube is $2^n/4$ for odd values of n as well as even.

Finding upper bounds on the forcing number is often interesting as well. It is easy to find perfect matchings on a $2m \times 2n$ torus with forcing number mn. It is an open question whether or not this is the maximum forcing number for all perfect matchings.

Conjecture. Except for the trivial case when $m=1$ or $n=1$, the forcing number of a perfect matching on a $2m \times 2n$ torus is at most mn.

The maximum forcing number for matchings in Q_n is unknown as well, but it can be shown that for sufficiently large n it must be near to the total number of edges in a perfect matching on Q_n

Proposition 3. *For sufficiently large n, there exist perfect matchings of Q_n with forcing number greater than $c2^{n-1}$ for any constant $c < 1$.*

The proof of this surprising result was found by Noga Alon [1].

Proof. Alon uses Van der Waerden's conjecture to show that

$$f(n) > \left(\frac{n}{e}\right)^{2^{n-1}},$$

where $f(n)$ denotes the number of perfect matchings of Q_n. On the other hand, if we assume that the maximum forcing number is at most $c2^{n-1}$, then there must be at least $f(n)$ distinct forcing sets of size $c2^{n-1}$ or smaller. By taking the number of subsets of size $c2^{n-1}$ or smaller from one partite class of Q_n and multiplying by the number of ways there are to select edges coming out of those vertices, we will count all matchings of size $c2^{n-1}$ or smaller, with some overcounting. The first term in this product is less than $2^{2^{n-1}}$, and the second is at most $n^{c2^{n-1}}$, so we have an upper bound on the number of perfect matchings of Q_n:

$$f(n) < 2^{2^{n-1}} n^{c2^{n-1}}.$$

Combining these inequalities and taking the 2^{n-1}th root of both sides, we get $n/e < 2n^c$, which will be false for sufficiently large n if $c < 1$. Therefore, the assumption that there are no perfect matchings with forcing number greater than $c2^{n-1}$ must have been false. $\square$

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